

EISENSTEIN SERIES AND MODULAR DIFFERENTIAL EQUATIONS

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ABSTRACT. The purpose of this paper is to solve various differential equations having Eisenstein series as coefficients using various tools and techniques. The solutions are given in terms of modular forms, modular functions and equivariant forms.

1. INTRODUCTION

In this paper we study differential equations of the form

$$y'' + \alpha E_4(z)y = 0,$$

where $E_4(z)$ is the weight 4 Eisenstein series, for different values of α yielding solutions of different kinds. These differential equations are interesting in many respects. The graded differential ring of modular forms and their derivatives is generated by the Eisenstein series E_2 , E_4 and E_6 , and the differential being $\frac{d}{2\pi i dz}$. Differentiating twice a homogeneous element of this ring would increase the weight by 4 and multiplying by E_4 would have the same effect. This justifies the form of the above second order linear differential equation with a particular interest towards the case where α is a rational multiple of π^2 (resulting from a double differentiation).

One particular second order linear differential equation that has drawn the attention of Klein [4], Hurwitz [2], Van der Pol [15] takes the form

$$(1.1) \quad y'' + \frac{\pi^2}{36} E_4(z)y = 0.$$

Van der Pol, [15], noticed that the Riccati equation attached to (1.1) is

$$(1.2) \quad \frac{6}{i\pi} u' + u^2 = E_4,$$

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and thanks to Ramanujan's identities, [9], one has $u = -E_2$ as a solution to (1.2). Thus one can recover the solution to (1.1) in terms of the classical discriminant function.

The purpose of this paper is to investigate differential equations similar to (1.2). Namely, we study the Riccati equation, or the corresponding linear ODE, of the form

$$(1.3) \quad \frac{k}{i\pi}u' + u^2 = E_4, \quad k = 1, \dots, 6.$$

and the corresponding second order ODE

$$(1.4) \quad y'' + \frac{\pi^2}{k^2}E_4(z)y = 0.$$

It turns out that in all but one case, modular functions arise as solutions. However, when $k = 1$, we will see that the solution has very special features leading to an example of the so-called equivariant forms developed by the authors in a forthcoming paper. In four cases, these equations can be transformed to Schwarzian differential equations which can be solved using techniques developed in a previous paper by one of the author [11].

It should be mentioned that differential equations with modular forms as coefficients appear in various contexts including in physics. Indeed, particular differential equations similar to (1.4) but with the presence of a first order term, namely

$$y'' - 4i\pi E_2(z)y' + \frac{44}{5}\pi^2 E_4(z)y = 0,$$

appear in [8] as a consequence of the character formula for the Virasoro model and the theory of vertex operator algebras. This leads to one of the Ramanujan identities. Surprisingly, the technique developed in [6, 11] in the level 5 case a few years before [8] can provide the same solution with purely number-theoretic arguments. Another similar equation

$$y'' - \frac{1}{3}i\pi E_2(z)y' + \frac{2}{3}\pi^2 E_4(z)y = 0$$

appears in [7] to classify rational conformal theories. On the other hand, an extensive study of some modular differential equations involving E_2 and E_2' as coefficients can be found in [3] of the form

$$(1.5) \quad y'' - \frac{k+1}{6}E_2(z)y' + \frac{k(k+1)}{12}E_2'(z)y = 0.$$

These equations are studied from the hypergeometric equations point of view. However, the only common feature with our modular differential equations is the fact that they are both homogeneous in the differential ring of quasi-modular forms. Moreover, the equation (1.5) can be symmetrized to have the form

$$(1.6) \quad \partial_{k+2}\partial_k y - \frac{k(k+1)}{144}E_4(z)y = 0$$

where $\partial_k y = y' - \frac{k}{12}E_2(z)y$ and using $12E_2'(z) = E_2^2 - E_4$. However, the equations (1.4) and (1.6) are different in their nature and in their solutions.

The paper is organized as follows: In Section 2, we recall the correspondence between second order linear ODE, the Riccati equations, and the Schwarz differential equation. In Section 3 we deal with the classical case of Van der Pol corresponding to the case $k = 6$ in (1.3). Section 4 deals with the cases $k = 2, 3, 4, 5$ in one shot using analytic properties of the Schwarzian derivative derived from a previous work [6], [11]. The most interesting case to us is when $k = 1$. In addition of providing the solution to the corresponding ODE, we find that this solution satisfies an intriguing equivariance property.

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2. LINEAR AND NON-LINEAR ODE'S

In this section, we recall some classical connections between three kinds of differential equations.

Let $R(z)$ be a meromorphic function on a domain D of the complex plane. The first equation under consideration is the second order linear differential equation of the form

$$(2.1) \quad y'' + \frac{1}{4}R(z)y = 0,$$

for which the space of local solutions is a two-dimensional vector space. The second differential equation is non-linear and has the form

$$(2.2) \quad \{f, z\} = R(z),$$

involving the Schwarz derivative

$$\{f, z\} = 2 \left(\frac{f''}{f'} \right)' - \left(\frac{f''}{f'} \right)^2.$$

If y_1 and y_2 are two linearly independent solutions to (2.1), then $f = y_1/y_2$ is a solution to (2.2). Moreover, a function g is a solution to (2.2) if and only if g is a non-constant linear fractional transform of f . Conversely, if f is a solution to (2.2) which is locally univalent, then $y_1 = f/\sqrt{f'}$ and $y_2 = 1/\sqrt{f'}$ are two linearly independent solutions to (2.1).

Meanwhile, if we set $u = f''/f'$, we see that (2.2) yields a Riccati equation

$$(2.3) \quad 2u' - u^2 = R(z).$$

Finally, one can go directly from solutions to (2.1) to solution to (2.3) by taking logarithmic derivatives.

In the remaining of this paper, we will provide solutions to certain differential equations by favoring the form that leads to a simple closed solution.

3. EISENSTEIN SERIES AS COEFFICIENTS OF ODE'S

The study of differential equations having modular forms as solutions or as coefficients has a long history going back to Dedekind, Schwarz, Klein, Poincaré, Hurwitz, Ramanujan, Van der Pol, Rankin and several others. The differential equations under consideration are Schrödinger type equations (or Sturm-Liouville equations) $y'' + Q(z)y = 0$ where $Q(z)$ is the potential. When $Q(z)$ satisfies some invariance properties, the Schrödinger equation leads to a variety of interesting theories. For instance, when the potential $Q(z)$ is periodic, then we have Mathieu's equation or Hill's equation widely studied in the last two centuries. When the potential is doubly periodic, we have Lamé equation which motivated extensive research leading to the modern theory of integrable systems. Also as it was mentioned in the introduction, differential equations with modular forms as coefficients have been extensively studied in physics and in relation with hypergeometric functions and quasimodular forms.

In this paper we consider equations with a potential that is automorphic or modular explicitly given by the Eisenstein series. We will propose solutions to these equations and show that their existence is very natural. Let us recall some definitions.

The Dedekind eta-function is defined on the upper half-plane $\mathcal{H} = \{z \in \mathbb{C}, \Im z > 0\}$ by

$$\eta(z) = q^{\frac{1}{24}} \prod_{n \geq 1} (1 - q^n), \quad q = e^{2\pi iz}, \quad z \in \mathcal{H}.$$

The discriminant Δ is defined by

$$\Delta(z) = \eta(z)^{24}.$$

The Eisenstein series $E_2(z)$, $E_4(z)$ and $E_6(z)$ are defined for $z \in \mathcal{H}$ by

$$E_2(z) = 1 - 24 \sum_{n \geq 1} \frac{nq^n}{1 - q^n} = 1 - 24 \sum_{n \geq 1} \sigma_1(n) q^n,$$

$$E_4(z) = 1 + 240 \sum_{n \geq 1} \frac{n^3 q^n}{1 - q^n} = 1 + 240 \sum_{n \geq 1} \sigma_3(n) q^n,$$

and

$$E_6(z) = 1 - 504 \sum_{n \geq 1} \frac{n^5 q^n}{1 - q^n} = 1 - 504 \sum_{n \geq 1} \sigma_5(n) q^n,$$

where $\sigma_k(n)$ is the sum of the k -th powers of all the positive divisors of n . The function E_4 and E_6 are modular forms for $\mathrm{SL}_2(\mathbb{Z})$ of respective weight 4 and 6. In [15], Van der Pol studied the equation

$$y'' + \frac{\pi^2}{36} E_4(z) y = 0,$$

and was later generalized to higher order differential equations in [14]. This equation appears also in [2] and in [4].

If we set

$$u = \frac{6}{i\pi} \frac{y'}{y},$$

then the differential equation satisfied by y transforms into a Riccati equation

$$\frac{6}{i\pi} u' + u^2 = E_4(z).$$

In the meantime, we have differential relations between Eisenstein series also known as the Ramanujan identities [9]

$$(3.1) \quad \frac{1}{2\pi i} \frac{d}{dz} E_2(z) = \frac{1}{12} (E_2^2 - E_4),$$

$$(3.2) \quad \frac{1}{2\pi i} \frac{d}{dz} E_4(z) = \frac{1}{3} (E_2 E_4 - E_6),$$

$$(3.3) \quad \frac{1}{2\pi i} \frac{d}{dz} E_6(z) = \frac{1}{2} (E_2 E_6 - E_4^2).$$

Using (3.1) and the fact that

$$(3.4) \quad E_2(z) = \frac{1}{2\pi i} \frac{\Delta'}{\Delta},$$

we have

Proposition 3.1. *The Riccati equation*

$$\frac{6}{i\pi} u' + u^2 = E_4(z)$$

has $u = -E_2(z)$ as a solution, and the corresponding linear differential equation

$$y'' + \frac{\pi^2}{36} E_4(z) y = 0$$

has as a solution

$$y = \Delta^{-\frac{1}{12}} = \eta^{-2}.$$

For the remainder of this paper, we study similar equations obtained by varying the coefficient of u' in the Riccati equation. Namely, we look at the following equations

$$(3.5) \quad \frac{k}{i\pi} u' + u^2 = E_4(z), \quad k = 1, \dots, 6.$$

While the case $k = 6$ has been the subject of this section, the cases $k = 2, \dots, 5$ will be treated in the next section, and the case $k = 1$ will be the subject of the following section.

4. THE MODULAR CASE

We follow a different approach in dealing with the cases $2 \leq k \leq 5$ in (3.5) by using the equivalent form involving the Schwarzian differential equations

$$(4.1) \quad \{f, z\} = \frac{k^2}{4\pi^2} E_4(z), \quad 2 \leq k \leq 5.$$

Suppose that f is a modular function for a finite index subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$, that is a meromorphic function defined on \mathcal{H} such that $f(\gamma \cdot z) = f(z)$ for $\gamma \in \Gamma$, $z \in \mathcal{H}$ and $\gamma \cdot z = \frac{az+b}{cz+d}$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and f extending to a meromorphic function on $\mathcal{H} \cup \{\infty\}$. The derivative f' is a weight 2 meromorphic modular form, and one can show that if g is a modular form of weight k , then $kgg'' - (k+1)g'^2$ is a modular form of weight $2k+4$. Applying this to f' , one obtains that $2f'f''' - 3f''^2$ is a modular form of weight 8, and if divided by f'^2 , it yields $\{f, z\}$ which is then a modular form of weight 4. A more interesting case occurs when the group Γ is of genus 0 in the sense that the compactification of the

Riemann surface $\Gamma \backslash \mathcal{H}$ is of genus 0. Indeed, if f is a Hauptmodul for Γ , that is a generator of the function field of the compact Riemann surface, then $\{f, z\}$ is actually a weight 4 modular form for exactly the normalizer of Γ in $\mathrm{SL}_2(\mathbb{R})$, [6]. Furthermore, the form $\{f, z\}$ has double poles at the elliptic fixed points of Γ (which are the critical points of f) and is holomorphic everywhere including at the cusps.

Recall that the space of holomorphic modular forms of weight 4 for $\mathrm{SL}_2(\mathbb{Z})$ is one-dimensional and is generated by E_4 . Thus, to be able to use the above analysis to get $\{f, z\}$ being a scalar multiple of E_4 , one should look for Γ being normal in $\mathrm{SL}_2(\mathbb{Z})$, having genus 0, and having no elliptic points. This is the case if one takes Γ to be one of the principal congruence groups $\Gamma(n)$, $2 \leq n \leq 5$. By choosing a Hauptmodul f for each such $\Gamma(n)$, one is able to identify exactly which multiple of E_4 is the modular form $\{f, z\}$. It turns out that this depends on the level n , which is also the cusp width at ∞ . Indeed, for $2 \leq n \leq 5$, a Hauptmodul f_n for $\Gamma(n)$ satisfies $f_n(z+n) = f_n(z)$, $z \in \mathcal{H}$, hence f_n has a Fourier expansion in $q = \exp(2\pi iz/n)$, and if one chooses f_n to vanish at ∞ , then we can take $f_n(z) = q + O(q)$ as $q \rightarrow \infty$. In this case, one can show directly from the expression of the Schwarz derivative that

$$\{f_n, z\} = \frac{4\pi^2}{n^2} + O(q) \text{ as } q \rightarrow \infty.$$

Notice that any other Hauptmodul will lead to the same conclusion since it would be a linear fractional transform of f_n and thus having the same Schwarz derivative. In conclusion, we have

Theorem 4.1. *Let f_n be any Hauptmodul for $\Gamma(n)$, $2 \leq n \leq 5$, then f_n is a solution to the equation*

$$\{f, z\} = \frac{4\pi^2}{n^2} E_4(z).$$

Furthermore, if we set

$$u_n = -\frac{n}{2\pi i} \frac{f_n''}{f_n'}, \quad 2 \leq n \leq 5,$$

then u_n is a solution of the Riccati equation

$$\frac{n}{i\pi} u' + u^2 = E_4.$$

To make this theorem effective, we shall provide explicit expressions for the above-mentioned Hauptmoduln. These expressions are classical and can be found for instance in [6] and [11].

Group	Hauptmodul
$\Gamma(2)$	$\left(\frac{\eta(2z)}{\eta(z/2)}\right)^8$
$\Gamma(3)$	$\left(\frac{\eta(3z)}{\eta(z/3)}\right)^3$
$\Gamma(4)$	$\frac{\eta(z/2)\eta(4z)^2}{\eta(z/4)^2\eta(2z)}$
$\Gamma(5)$	$q^{1/5} \prod_{n \geq 1} (1 - q^n)^{(\frac{n}{5})}$

(\div) is the Legendre symbol

This completes the study of the equations (3.5) for the cases $2 \leq k \leq 5$, and the solutions are expressed in terms of modular functions.

Remark 4.2. The case $k = 6$ settled in Proposition 3.1 cannot use the method of this section since $\Gamma(6)$ is not of genus 0. The same applies to the case $k = 1$ since $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$ has elliptic fixed points. This case shall be studied in the next section.

5. THE EQUIVARIANT CASE

We now focus on the case $k = 1$ in (3.5) or its equivalent form in terms of the second order linear differential equation. We have

Theorem 5.1. *The function $y = E'_4 \Delta^{-1/2}$ is a solution to*

$$(5.1) \quad y'' + \pi^2 E_4 y = 0.$$

Proof. Using (3.4) one has $\Delta' = 2\pi i \Delta E_2$, and using (3.1), one finds that

$$\Delta'' = -\frac{13\pi^2}{3} \Delta E_2^2 + \frac{\pi^2}{3} \Delta E_4.$$

It follows that for $y = E'_4 \Delta^{-1/2}$ we have

$$y'' + \pi^2 E_4 y = \Delta^{-\frac{1}{2}} [E_4''' - 2\pi i E_4'' E_2 - \frac{5\pi^2}{6} E_4' E_2^2 + \frac{5\pi^2}{6} E_4' E_4].$$

Meanwhile, since the differential ring of modular forms and their derivatives is $(\mathbb{C}[E_2, E_4, E_6], \frac{1}{2\pi i} \frac{d}{dz})$, one is able to express every term algebraically by means of the generators E_2 , E_4 and E_6 . Indeed, using Ramanujan identities (3.1), (3.2) and (3.3), one has

$$E_4' = \frac{2\pi i}{3} (E_2 E_4 - E_6),$$

$$E_4'' = -\frac{5\pi^2}{9} (E_2^2 E_4 - 2E_2 E_6 + E_4^2),$$

and

$$E_4''' = -\frac{5\pi^3 i}{9} (E_4 E_2^3 + 3E_4^2 E_2 - 3E_2^2 E_6 - E_4 E_6).$$

Now, substituting into the expression of $y'' + \pi^2 E_4 y$ shows it is identically equal to 0.

□

We mentioned in the previous section that if g is a modular form of weight k , then $(k+1)g'^2 - kgg''$ is a modular form of weight $2k+4$. If $g = E_4$, then $5E_4'^2 - 4E_4 E_4''$ is a holomorphic modular form of weight 12, which clearly vanishes at ∞ and thus it is a multiple of Δ . Investigating the leading coefficient yields

$$5E_4'^2 - 4E_4 E_4'' = -3840\pi^2 \Delta.$$

While $y = (1/i\pi\sqrt{3840})E_4' \Delta^{-1/2}$ is still a solution of (5.1), we then would like to find a function f such that $1/\sqrt{f'} = y$. This, according to Section 2, will provide us with a solution to the corresponding Schwarzian equation $\{f, z\} = 4\pi^2 E_4(z)$. The function would satisfy

$$\begin{aligned} f'(z) &= \frac{1}{y^2} \\ &= \frac{-3840\pi^2 \Delta}{E_4'^2} \\ &= \frac{5E_4'^2 - 4E_4 E_4''}{E_4'^2} \\ &= 1 + 4 \left(\frac{E_4}{E_4'} \right)'. \end{aligned}$$

Thus, we have the following

Theorem 5.2. *The function defined on \mathcal{H} by*

$$(5.2) \quad f(z) = z + 4 \frac{E_4}{E_4'}$$

is a solution to

$$\{f, z\} = 4\pi^2 E_4(z)$$

As a consequence, we have

Corollary 5.3. *The function*

$$u = \frac{1}{i\pi} \frac{E_4''}{E_4'} - E_2$$

is a solution to the Riccati equation

$$\frac{1}{i\pi} u' + u^2 = E_4$$

Unlike the previous section, the solution (5.2) is not a modular function and in fact it is not invariant under any non-identity element of $SL_2(\mathbb{Z})$. More precisely it has the following surprising equivariance properties.

Theorem 5.4. *The function f defined on \mathcal{H} by (5.2) satisfies*

$$(5.3) \quad f\left(\frac{az+b}{cz+d}\right) = \frac{af(z)+b}{cf(z)+d}, \quad z \in \mathcal{H}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

Proof. To show that $f(\gamma \cdot z) = \gamma \cdot f(z)$ holds for all $\gamma \in SL_2(\mathbb{Z})$, it suffices to prove that it holds for the two generators $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ of $SL_2(\mathbb{Z})$. It is clear that $f(z+1) = f(z) + 1$ and we only need to check that $f(-1/z) = -1/f(z)$. Since $E_4(-1/z) = z^4 E_4(z)$ we have $E_4'(-1/z) = 4z^5 E_4(z) + z^6 E_4'(z)$. Hence

$$\begin{aligned} f\left(\frac{-1}{z}\right) &= \frac{-1}{z} + \frac{4E_4(z)}{4zE_4(z) + z^2E_4'(z)} \\ &= \frac{-E_4'}{4E_4 + zE_4'} \\ &= \frac{-1}{z + 4\frac{E_4}{E_4'}} \\ &= \frac{-1}{f(z)}. \end{aligned}$$

This concludes the proof. \square

The function f above is an example of the so-called *equivariant forms* developed by the authors in a forthcoming paper [12] regarding meromorphic functions defined on the upper half-plane and commuting with the action of a discrete group, see also [1, 13]. In fact, for a subgroup

of the modular group, a subclass of these equivariant forms consists of functions f defined by

$$f(z) = z + k \frac{g(z)}{g'(z)},$$

where $g(z)$ is a modular form of weight k , [13]. What makes the equivariance property work is that the function

$$F(z) = \frac{k}{f(z) - z} = \frac{g'(z)}{g(z)}$$

transforms as

$$(cz + d)^{-2} F\left(\frac{az + b}{cz + d}\right) = F(z) + \frac{kc}{cz + d},$$

which, in the terms of M. Knopp, [5], makes $F(z)$ a modular integral with period function $\frac{kc}{cz+d}$, or a quasi-modular form in the language of D. Zagier. Further connections between these subjects and equivariant forms will be made explicit in [12].

Proposition 5.5. *The derivative of f is given by*

$$f'(z) = -3840\pi^2 \frac{\Delta}{E_4'^2}.$$

Proof. We have

$$f'(z) = \frac{5E_4'^2 - 4E_4E_4''}{E_4'^2}.$$

It is a known fact that if f is a modular form of weight k , then $(k+1)f'^2 - kff''$ is a modular form of weight $2k+4$. Hence $5E_4'^2 - 4E_4E_4''$ is a modular form of weight 12 vanishing at ∞ . It turns out that $5E_4'^2 - 4E_4E_4'' = -3840\Delta$, and the proposition follows. \square

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